# EC202- PS1 Solutions

**1. One of the most commonly used functions in microeconomics is the Cobb-Douglas function**

$$y=(x_1)^{\alpha}\ (x_2)^{\beta}$$

**where α , β are positive and <1 parameters.**

**i. Show that this function is quasi-concave.**

<u>Answer:</u>

$f_1 = \ =\alpha x_1^{\alpha-1}\ x_2^{\beta}$

$f_2 = \ =\beta x_1^{\alpha}\ x_2^{\beta-1}$

$f_{11} = \ =\alpha(\alpha-1)x_1^{\alpha-2}\ x_2^{\beta} \quad <0$

$f_{22} = \ =\beta(\beta-1)x_1^{\alpha}\ x_2^{\beta-2}<0$

$f_{12} = f_{21}\ = \alpha\beta x_1^{\alpha-1}\ x_2^{\beta-1}>0$

***Remember that the condition for quasi-concavity is :***

$f_{11}\ f_2^2 - 2f_{12}\ f_1\ f_2\ + f_{22}\ f_1^2\ <0$

***which is clearly satisfied since all terms above are negative.***

**ii.**

**Show that if α + β>1 than the Cobb-Douglas function is not concave (which shows that not all quasi concave functions are concave).**

<u>Answer:</u>

***Recall that a function is concave iff:***

$f_{11}\ f_{22} f_2^2 - f_{12}^{\ 2} >0$

$f_{11}\ f_{22} f_2^2 - f_{12}^{\ 2} = \alpha\beta\,(1-\beta-\alpha)\ x_1^{2\alpha-2}\ x_2^{\beta 2-2}$ ***which is negative for*** $\beta+\alpha>1.$

**2. Proof of the Envelop Theorem (this may help to develop some intuitions!!).**

**Suppose we wish to maximise a function of 2 variables and that the value of this function also depends on a parameter, *a* : $f(x_1,x_2,a)$. This maximisation problem is subject to a constraint that can be written as: $g(x_1,x_2,a)=0$.**

i. **Write down the Lagrangian expression of the FOC for this problem.**

Answer:

$L(x_1,x_2,a)= f(x_1,x_2,a)+ \lambda\ g(x_1,x_2,a)$

*The FOCs are:*

$L_1 = f_1 \ + \lambda g_1=0$

$L_2 = f_2 \ + \lambda g_2=0$

$L_\lambda = g = 0$

ii. **Sum the two FOCs involving the x s.**

iii. **Now differentiate the above sum with respect to a- this show how the x s must change as a changes while requiring that the FOCs continue to hold**

Answer:

*Multiplication of each of the FOC by the appropriate derivatives yields:*

$$f_1\frac{dx1}{da} + f_2\frac{dx2}{da} + \lambda(g_1\frac{dx1}{da} + g_1\frac{dx2}{da})=0$$

iv. **As shown in the notes (LN 1) both the objective function and the constraint can be stated as a function of a: *$f(x_1(a),x_2(a),a)$; $g(x_1(a),x_2(a),a)$*. Differentiate the first of these with respect to a . This shows how the value of the objective changes as a changes while keeping the x's at their optimal values. You should have terms that involves the x's and a single term in ∂f/∂a.**

Answer:

***The optimal vaule of f is given by $f(x_1(a),x_2(a),a)$. Differentiation of this with repect to a shows how its optimal value changes with a:***

$$\frac{df*}{da} = f1\frac{dx1}{da} + f2\frac{dx2}{da} + fa$$

v. **Now differentiate the constraint as formulated in part (iv) with respect to a. You should have terms in the x's and a single term in ∂g/∂a.**

Answer:

***Differentiaition of the constraint $g(x_1(a),x_2(a),a).=0$ yiels:***

$$\frac{dg}{da} = 0 = g1\frac{dx1}{da} + g2\frac{dx2}{da} + ga$$

vi. **Multiply the results from part (v) by $\lambda$ (the Lagrange multiplier) and use this together with the FOCs from part (iii) to substitute into the derivative from part (iv). You should be able to show that:**

$$\frac{df(x_1(a), x_2(a), a)}{da} = \frac{\partial f}{\partial a} + \lambda\frac{\partial g}{\partial a}$$

**Which is just the partial derivative of the Lagrangean expression when all the x's are at their optimal values. This proves the theorem. Explain intuitively this result.**

**Answer:**

***Multiplying the results from part (v) and using parts (ii) and (iii) yields:***

$$\frac{\partial f}{\partial a} = fa + \lambda ga = La$$

**Which proves the theorem.**

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**3. Showing convexity of indifference curves. Calculation of the MRS for specific utility function is often a good shortcut for showing convexity of ICs and the process can be much easier than applying the definition of quasi concavity (although is much more difficult to generalise to more than 2 goods!!). Show whether or not the following utility functions are represented by convex ICs**

**Answer:**

***i.*** ***A shortcut that can simplify the algebra is to take the log of this function. Because taking the logs is order preserving will not alter the MRS to be calculated***

1. $U(x,y) = \sqrt{x \cdot y}$

Let $U^*(x,y) = \ln[U(x,y)] = 0.5\ln x + 0.5\ln y$

$$MRS = \frac{\partial U^* / \partial x}{\partial U^* / \partial y} = \frac{y}{x}$$

- ***MRS is diminishing as x increases and y decreases***
- ***Therefore, the indifference curves are convex***

**ii.** **There is no obvious shortcut here.**

2. $U(x,y) = x + xy + y$

$$MRS = \frac{\partial U / \partial x}{\partial U / \partial y} = \frac{1+y}{1+x}$$

- *MRS is diminishing as x increases and y decreases*
- *Therefore, the indifference curves are convex*

iii. **For this example it is convenient to use the following transformation.**

3. $U(x,y)=\sqrt{x^2+y^2}$

Let $U^*(x,y)=[U(x,y)]^2=x^2+y^2$

$$MRS=\frac{\partial U^*/\partial x}{\partial U^*/\partial y}=\frac{x}{y}$$

- ***As x increases and y decreases, the MRS increases!***
- ***The indifference curves are concave, not convex***
- ***This is not a quasi-concave function***

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**4. Consider a Cobb Douglas utility function U(x,y)= $x^{\alpha}+y^{\beta}$.**

i. **Calculate the MRS. Does this result depend on whether α + β=1? Does this sum have any relevance to the theory of choice?**

**Answer**

$$MRS=\frac{\partial U}{\partial x}/\frac{\partial U}{\partial y}=(\alpha x_1^{\alpha-1}x_2^{\beta})/(\beta x_1^{\alpha}x_2^{\beta-1})=\frac{\alpha}{\beta}\frac{y}{x}$$

***The results does not depend on α + β. No relevance to the theory of choice (but has relevance for production theory).***

ii. **For commodity bundles for which x=y, how does the MRS depend on the values of α and β? Give an intuition for why if α > β, MRS>1. Illustrate your argument with a graph.**

**Answer**

*The mathematics follow directly from (i). If α > β the individual values x relatively more highly. Hence MRS>1 for x=y.*

iii. **Suppose an individual obtains utility only from amounts of x and y that exceed minimal subsistence levels given by $x_0$ , $y_0$. In this case**

$$U(x,y)= (x - x_0)^{\alpha}+(y - y_0)^{\beta}.$$

**Is this function homothetic?**

**Answer**

***The function is homothetic in $(x - x_0)$ and $(y - y_0)$, but not in x and y.***

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**5**

i. **Show that the CES function**

$$\alpha\frac{x^{\delta}}{\delta}+\beta\frac{y^{\delta}}{\delta}$$

**is homothetic. How does the MRS depends on the ratio y/x?**

**Answer**

***MRS***$=\frac{\partial U}{\partial x}\Big/\frac{\partial U}{\partial y}=\frac{\alpha x^{\delta-1}}{\beta\frac{y^{\delta-1}}{\delta}}=\frac{\alpha}{\beta}(x/y)^{\delta-1}$, ***so this function is homothetic.***

ii. **Show that the MRS is strictly diminishing for all values of** $\delta<1$.

**Answer**

$\frac{\partial MRS}{\partial x}=(\delta-1)\frac{\alpha}{\beta}y^{1-\delta}x^{\delta-2}$ ***This is negative iff*** $\delta<1$.

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6. **Consider the function U(x,y)= x+ln y. Note that the function is linear in x and non-linear in y, for this reason it is referred as quasi-linear utility function.**

i. **Find the MRS and interpret the results.**

Answer

*MRS=y*

ii. **Show that the function is quasi concave.**

Answer

*Check:* $f_{11} f_2^2 - 2f_{12} f_1 f_2 + f_{22} f_1^2 < 0$

*We have*

$f_x = f_1 = 1$

$f_y = f_2 = 1/y$

$f_{11} = 0$

$f_{22} = -1/y^2$

$f_{12} = 0$

*so* $f_{11} f_2^2 - 2f_{12} f_1 f_2 + f_{22} f_1^2 = 0 + 0 - 1/y^2 = -1/y^2$

*which is negative for positive ys.*

iii. **Find the equation for the IC for this function.**

Answer

$$y = e^{C-x}$$

iv. **Compare marginal utility of x and y. How do you How do you interpret this function?**

Answer

*Since the MU of x is a constant at 1, while that of y is decreasing (in the form 1/y) we would expect consumers to shift more towards x and*

***away from y when they get to buy more goods to increase utility due to an income raise. This is because consumers max utility. They are better off with higher MU.***

v. **Consider how the utilities changes as the quantities of the two goods increase, describe some situations where this function might be useful.**

**Answer**

***This function is usually used to describe the consumption of one commodity wrt all other commodities, So ln y could represent the singular commodity while x all the other goods consumed.***